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On the lowest eigenvalue of the Laplacian with Neumann boundary condition at a small obstacle

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Abstract

We study the lowest eigenvalue $\lambda_1(\varepsilon)$ of the Laplacian $-\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, from which a small compact set $K_\varepsilon \subset B_\varepsilon$ has been deleted, imposing Dirichlet boundary conditions along $\partial\Omega$ and Neumann boundary conditions on ∂K_ε. We are mainly interested in results that require minimal regularity of ∂K_ε expressed in terms of a Poincaré condition for the domains $\Omega\backslash\varepsilon^{-1}K_\varepsilon$. We then show that $\lambda_1(\varepsilon)$ converges to Λ_1, the first Dirichlet eigenvalue of Ω, as $\varepsilon \to 0$. Assuming some more regularity we also obtain asymptotic bounds on $\lambda_1(\varepsilon) - \Lambda_1$, for ε small, where we employ an idea of [Burenkov and Davies, J. Differential Equations 186 (2002) 485–508].

MSC: 35P15; 35J20

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0. Introduction

It is a common expectation that small perturbations of the physical situation lead only to a small change of the spectrum. In the case of domain perturbations this is largely true for Dirichlet boundary conditions while the Neumann case is more delicate. In fact, imposing Neumann conditions at the boundary of an arbitrarily small hole in a bounded domain Ω may produce essential spectrum equal to any preassigned

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closed set $S \subset [0, \infty)$; cf. [13,14]. Any attempt at a numerical computation of eigenvalues in the Neumann case has to take into account the difficulties that stem from the irregularities of the boundary, as is well known [3].

In the present paper, we study the eigenvalue problem for the Laplacian $-\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, with a hole K_ε where $K_\varepsilon \subset B_\varepsilon \subset \Omega$ is compact, $\varepsilon > 0$ is small, and $\Omega \backslash K_\varepsilon$ is connected; cf. Section 2 for complete assumptions. We consider mixed boundary conditions: Dirichlet boundary conditions on $\partial\Omega$ and Neumann boundary conditions on ∂K_ε. The basic setting corresponds to a simple model for a membrane with a small hole or tear. There are rather complete results for smooth boundary ∂K_ε with $K_\varepsilon = \varepsilon K_1$ [24–26,21] where analytic methods (e.g., expansions of the Green's function [11]) may be employed to obtain asymptotic expansions or asymptotic estimates for the boundary value problem and for the eigenvalue problem.

In contrast, we focus here on the case where the boundary of K_ε is not smooth. Due to the lack of smoothness we mainly have to rely on variational methods (Rayleigh–Ritz or Weyl's min–max principle). While variational methods effortlessly yield *upper bounds* for the lowest eigenvalue $\lambda_1(\varepsilon)$, corresponding *lower bounds* constitute a serious problem in the Neumann case due to the difficulty of extending eigenfunctions from $\Omega \backslash K_\varepsilon$ to all of Ω. Under minimal assumptions as to the regularity of ∂K_ε it is shown that $\lambda_1(\varepsilon)$ converges to Λ_1, the first Dirichlet eigenvalue of Ω; cf. Theorem 2.4. The proof combines min–max arguments with scaling and an estimate for the eigenvalue problem on an annulus with mixed Dirichlet and Neumann boundary conditions. Our methods have some potential for generalizations like replacing the Laplacian $-\Delta$ with a general elliptic divergence type operator $-\sum \partial_j a_{ij}(x)\partial_i$.

In the second half of our paper we then implement the idea of [3] of using a smooth comparison problem to produce lower bounds; cf. Proposition 4.1. For this approach to work we need good estimates for the smooth problem and an a priori bound on the eigenfunctions u_ε in L_p-norm, for some $p > 2$. The best estimates will follow from a bound in the maximum-norm; in this case (cf. Corollary 4.4) we obtain

$$|\lambda_1(\varepsilon) - \Lambda_1| \leqslant c\varepsilon^d, \quad 0 < \varepsilon < \varepsilon_0. \tag{0.1}$$

The paper is organized as follows. In Section 1, we introduce notation and some basic results. In particular, we define the Sobolev spaces required for the definition of the Laplacian on $\Omega \backslash K_\varepsilon$ with mixed boundary conditions via quadratic forms; this operator is denoted as H_ε in the sequel.

In Section 2, we discuss continuity properties of the first eigenvalue under minimal regularity assumptions. We give examples where essential spectrum is present. The variational methods used in this section also yield a basic upper estimate $\lambda_1(\varepsilon) \leqslant \Lambda_1 + c\,|K_\varepsilon|$, for $\varepsilon > 0$ small.

In Section 3, we prove the analogue of (0.1) for the smooth comparison problem with $K_\varepsilon = \overline{B}_\varepsilon$. While the required estimate is available from [24–26] in the case of $\mathbb{R}^2$ we give an independent and simple proof based on an expansion of the eigenfunctions u_ε in terms of Bessel functions in an annulus $B_\delta \backslash B_\varepsilon$, for $\delta > 0$ fixed and $0 < \varepsilon < \delta$. We then consider the harmonic extension of u_ε into the hole K_ε, a standard method [28,2].

In Section 4 we "interpolate" the case studied in Section 3 as a smooth comparison problem to produce lower bounds on $\lambda_1(\varepsilon)$ that depend on an a priori L_p-estimate on u_ε for some $p > 2$. We finally discuss two typical situations where L_p-estimates for the eigenfunctions u_ε can be derived. First, if ∂K is Lipshitz and $K_\varepsilon = \varepsilon K_1$, the Sobolev Extension Theorem may be combined with the Sobolev Embedding Theorem to yield an estimate for u_ε in L_p-norm, p a Sobolev exponent, which leads to

$$\Lambda_1 - c\varepsilon^2 \leqslant \lambda_1(\varepsilon) \leqslant \Lambda_1 + c'\varepsilon^d, \quad 0 < \varepsilon < \varepsilon_0. \tag{0.2}$$

Second, if K_ε is (roughly speaking) a limit of star-shaped regular domains, one can give an a priori bound for the maximum-norm of u_ε by a simple comparison argument. This yields an estimate as in (0.1).

For simplicity and clarity of the argument we have preferred to restrict our discussion to the lowest eigenvalue $\lambda_1(\varepsilon)$ in this paper; we expect that most of our results can be extended to the higher eigenvalues λ_k, $k \geqslant 2$, of H_ε.

We conclude the introduction with some remarks on related work in the literature. The case of smooth obstacles $K_\varepsilon = \varepsilon K_1$ with Dirichlet boundary conditions has been analyzed in great detail (cf., e.g., [21,24]). A rather complete analysis of the smooth Neumann problem has been given in [24–26] and [21]. A main tool in Ozawa's approach are expansion of the Green's function (or Neumann function) in the vein of the classical paper [11] which also gives a Hadamard type formula expressing the derivative of Neumann eigenvalues with respect to variations of the domain by a boundary integral. In Chapter 9 of [21], the asymptotics of the first eigenvalue and the associated eigenfunction are derived by direct expansion of the relevant quantities. Some interest in (many) small holes with Dirichlet or Neumann boundary conditions comes from the celebrated "crushed ice problem" [28,32,24,36]. In particular, in view of [28] and homogenization theory it appears that the first eigenvalue is of special interest. The basic situation studied in our paper is also related to the study of the Laplace Beltrami operator on manifolds with thin attachments ("handles") where lower bounds for the eigenvalues pose similar difficulties (cf., e.g., [2,4,27]). Finally, scattering for an exterior domain $\mathbb{R}^d \setminus K$, with K compact and Neumann boundary conditions along ∂K, has been considered in [15] for the case where ∂K creates a second scattering channel.

1. Notation and preliminaries

For $x_0 \in \mathbb{R}^d$ and $r > 0$, we write $B_r(x_0) = \{x \in \mathbb{R}^d;\ |x - x_0| < r\}$, and $B_r = B_r(0)$. For $G \subset \mathbb{R}^d$, open, we denote the Lebesgue spaces as $L_p(G)$, $1 \leqslant p \leqslant \infty$, with norm $\|\cdot\|_p$. For $p = 2$ we obtain the Hilbert space $L_2(G)$ with norm $\|\cdot\| := \|\cdot\|_2$ and scalar product $\langle \cdot\,,\,\cdot \rangle$. For $M \subset \mathbb{R}^d$ a Borel set, $|M|$ denotes the Lebesgue measure of M.

For a self-adjoint operator T acting in a Hilbert space $\mathscr{H}$ we use the usual definitions and notation ([6,17,29]) for domain, range, spectrum, essential spectrum and discrete spectrum of T. We also use the standard definition for Sobolev spaces

$$\mathscr{H}^{1,p}(G) = \{f \in L_p(G);\ \nabla f \in L_p(G)^d\}, \quad 1 \leqslant p < \infty. \tag{1.1}$$

In particular, $\mathscr{H}^{1,p}_\circ(G)$ is the closure of $C_c^\infty(G)$ in the norm of $\mathscr{H}^{1,p}(G)$, given by

$$\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p, \quad 1 \leqslant p < \infty. \tag{1.2}$$

For $p = 2$, $\mathscr{H}^1(G) := \mathscr{H}^{1,2}(G)$ and $\mathscr{H}^1_\circ(G) := \mathscr{H}^{1,2}_\circ(G)$ are Hilbert spaces. For basic results on Sobolev spaces we refer to [1,8,20,12].

It is a well-known fact (cf., e.g., [5]), that we may delete a single point from G without affecting either of the Sobolev spaces $\mathscr{H}^1_\circ(G)$ or $\mathscr{H}^1(G)$:

Lemma 1.1. *Let $d \geqslant 2$, $G \subset \mathbb{R}^d$ be open, and let $x_0 \in G$. Then $C_c^\infty(G \setminus \{x_0\})$ is dense in $C_c^\infty(G)$ in the topology of $\mathscr{H}^1_\circ(G)$. Furthermore, $\mathscr{H}^1_\circ(G \setminus \{x_0\}) = \mathscr{H}^1_\circ(G)$ and $\mathscr{H}^1(G \setminus \{x_0\}) = \mathscr{H}^1(G)$. (More precisely, the restriction map from G to $G \setminus \{x_0\}$ establishes an isomorphism between $\mathscr{H}^1_\circ(G)$ and $\mathscr{H}^1_\circ(G \setminus \{x_0\})$.)*

The Dirichlet Laplacian on an open set $G \subset \mathbb{R}^d$, denoted by H^G, is defined as the unique self-adjoint, non-negative operator acting in $L_2(G)$ that satisfies $D(H^G) \subset \mathscr{H}^1_\circ(G)$ and

$$\langle H^G u\,,\, v\rangle = \langle \nabla u\,,\, \nabla v\rangle, \quad u \in D(H^G), \;\; v \in \mathscr{H}^1_\circ(G), \tag{1.3}$$

cf. [17, Theorem VI-2.1]. For smooth $u \in D(H^G)$ we have $H^G u = -\Delta u$. Let now $G \subset \mathbb{R}^d$ be a bounded domain (i.e., G is bounded, open and connected). Then H^G is positive, self-adjoint, with compact resolvent, and the spectrum of H^G consists of a sequence of eigenvalues $(\Lambda_k^G)_{k\in\mathbb{N}}$, repeated according to their respective multiplicities,

$$0 < \Lambda_1^G < \Lambda_2^G \leqslant \Lambda_3^G \leqslant \cdots, \tag{1.4}$$

tending to ∞ as $k \to \infty$; the first eigenvalue Λ_1^G is non-degenerate as G is connected. These eigenvalues can be characterized by the Rayleigh–Ritz variational principle (or Weyl variational formula), cf. [30,6]. For example, we have

$$\Lambda_1^G = \inf\{\|\nabla u\|^2;\; u \in \mathscr{H}^1_\circ(G), \|u\| = 1\}. \tag{1.5}$$

A simple, but important consequence is "domain monotonicity" of the Dirichlet eigenvalues: if $G \subset G'$, we have $\Lambda_k^{G'} \leqslant \Lambda_k^G$, for all $k \in \mathbb{N}$. We will need the following classic result which is an easy consequence of Lemma 1.1 and variational arguments (min–max principle).

Lemma 1.2. *Let $G \subset \mathbb{R}^d$ be a bounded domain, let $x_0 \in G$ and let $\Lambda_j^{G_\varepsilon}$ denote the eigenvalues (ordered according to min–max) of the Dirichlet Laplacian on $G_\varepsilon = G\setminus\overline{B_\varepsilon(x_0)}$. Then $\Lambda_j^{G_\varepsilon} \to \Lambda_j^G$, as $\varepsilon \to 0$, for all $j \in \mathbb{N}$.*

We next turn to Neumann and mixed boundary conditions. Replacing the Sobolev space $\mathscr{H}^1_\circ(G)$ in (1.3) with $\mathscr{H}^1(G)$, we obtain the Neumann Laplacian of an open domain $G \subset \mathbb{R}^d$. Even if G is bounded it is possible that all min–max values of the Neumann Laplacian are zero or, put differently, that 0 belongs to the essential spectrum of the Neumann Laplacian. The Neumann min–max values decrease if a (closed) set of Lebesgue measure zero is deleted from G.

The following notation and definitions for mixed boundary conditions are specific to our problem. Let $d \geqslant 2$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain containing the ball B_1 and let $K_\varepsilon \subset B_{\zeta\varepsilon}$ be compact, for $0 < \varepsilon \leqslant 1$, where $\zeta \in (0, 1)$ is independent of ε. We then define

$$\Omega_\varepsilon = \Omega\setminus K_\varepsilon, \quad 0 < \varepsilon \leqslant 1. \tag{1.6}$$

We will impose *mixed* boundary conditions for Ω_ε, namely Dirichlet boundary conditions on $\partial\Omega$ and Neumann boundary conditions on ∂K_ε. For this purpose we need an appropriate Sobolev space. Let $\varphi \in C_c^\infty(B_1)$ satisfy $\varphi(x) = 1$ for all $x \in B_\zeta$, and let $\psi = 1 - \varphi$. We then define

$$\mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon) = \{u \in \mathscr{H}^1(\Omega_\varepsilon);\; \psi u \in \mathscr{H}^1_\circ(\Omega)\}, \quad 0 < \varepsilon \leqslant 1. \tag{1.7}$$

It is easy to see that the space $\mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon)$ is independent of the choice of φ. We then let H_ε denote the unique self-adjoint operator with domain $D(H_\varepsilon) \subset \mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon)$ and

$$\langle H_\varepsilon u\,,\, v\rangle = \langle \nabla u\,,\, \nabla v\rangle, \quad u \in D(H_\varepsilon), \;\; v \in \mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon). \tag{1.8}$$

In the special case where K_ε is a single point, $K_\varepsilon = \{0\}$, we have again

$$\mathscr{H}^1_{\mathrm{DN}}(\Omega\backslash\{0\}) = \mathscr{H}^1_\circ(\Omega). \tag{1.9}$$

The min–max values according to [30, Theorem XIII.2] of H_ε are denoted as

$$0 \leqslant \lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \cdots, \quad 0 < \varepsilon \leqslant 1. \tag{1.10}$$

So far, it is quite possible that $\inf \sigma_{\mathrm{ess}}(H_\varepsilon) = 0$ in which case we would have $\lambda_j(\varepsilon) = 0$, for all $j \in \mathbb{N}$. Any $\lambda_j(\varepsilon)$ that lies below the infimum of $\sigma_{\mathrm{ess}}(H_\varepsilon)$ is a discrete eigenvalue of H_ε. The $\lambda_j(.)$ are monotonic with respect to variations of Ω.

Occasionally, we will need to specify the sets Ω and K_ε explicitly in the notation. Let Ω' and K'_ε satisfy the same assumptions as Ω and K_ε. We then let $H(\Omega', K'_\varepsilon)$ denote the Laplacian on $\Omega'\backslash K'_\varepsilon$ with mixed Dirichlet and Neumann boundary conditions as above; its min–max values are written as $\lambda_k(\Omega', K'_\varepsilon) \geqslant 0$. In particular, $H_\varepsilon = H(\Omega, K_\varepsilon)$ and $\lambda_1(\varepsilon) = \lambda_1(\Omega, K_\varepsilon)$.

2. Continuity of the lowest eigenvalue

In this section, we show that the lowest eigenvalue $\lambda_1(\varepsilon)$ of the mixed Dirichlet–Neumann problem on $\Omega_\varepsilon = \Omega\backslash K_\varepsilon$ converges to Λ_1, the first Dirichlet eigenvalue of Ω, under minimal regularity assumptions. Furthermore, we determine the dependence of the essential spectrum of H_ε on ε in the special case where $K_\varepsilon = \varepsilon K_1$, and we obtain a general upper bound of variational type for $\lambda_1(\varepsilon)$.

We will be working with the following assumptions concerning the sets Ω and K_ε, $0 < \varepsilon \leqslant 1$.

Assumption I

(a) $\Omega \subset \mathbb{R}^d$ is a domain with $B_1 \subset \Omega \subset B_R$, for some $R \geqslant 1$.
(b) For $0 < \varepsilon \leqslant 1$, we are given compact sets $K_\varepsilon \subset B_{\zeta\varepsilon}$, where $\zeta \in (0, 1)$ is independent of ε.
(c) $\Omega\backslash K_\varepsilon$ is connected, for $0 < \varepsilon \leqslant 1$.

Note that there is a certain arbitrariness in choosing the radii of the balls in Assumption I. We could as well have required $B_2 \subset \Omega$ and $K_\varepsilon \subset \overline{B}_\varepsilon$. In Section 3, we will in fact use this liberty and deal with $K_\varepsilon = \overline{B}_\varepsilon$ without rescaling.

Assumption II. For $0 < \varepsilon \leqslant 1$, the open domains $B_1\backslash\varepsilon^{-1}K_\varepsilon$ satisfy a *uniform Poincaré-type inequality*

$$\|u\|^2 \leqslant C_P \|\nabla u\|^2, \quad u \in \mathscr{H}^1_{\mathrm{DN}}(B_1\backslash\varepsilon^{-1}K_\varepsilon), \tag{2.1}$$

with a constant C_P independent of $0 < \varepsilon \leqslant 1$.

In the following remarks we always assume that Assumption I holds.

Remarks

(a) Assumption II is obviously equivalent to the existence of a positive constant $C_0 > 0$ such that

$$\lambda_1(B_1, \varepsilon^{-1}K_\varepsilon) \geqslant C_0, \quad 0 < \varepsilon \leqslant 1, \tag{2.2}$$

where $\lambda_1(B_1, \varepsilon^{-1}K_\varepsilon)$ denotes the lowest min–max value for the Laplacian of $B_1 \backslash \varepsilon^{-1}K_\varepsilon$, with Dirichlet boundary condition on ∂B_1 and Neumann boundary condition on $\varepsilon^{-1}\partial K_\varepsilon$.

(b) Assumption II implies a Poincaré inequality for the Sobolev space $\mathscr{H}^1(B_1 \backslash \varepsilon^{-1}K_\varepsilon)$,

$$\|u - \overline{u}\|^2 \leqslant C'_P \|\nabla u\|^2, \quad u \in \mathscr{H}^1(B_1 \backslash \varepsilon^{-1}K_\varepsilon), \tag{2.3}$$

where $C'_P \geqslant 0$ is a constant and $\overline{u} = |B_1 \backslash \varepsilon^{-1}K_\varepsilon|^{-1} \int u \, dx$ denotes the mean value of u over $B_1 \backslash \varepsilon^{-1}K_\varepsilon$. We defer the proof until the end of this section. Note that (2.3) deals with the *second Neumann* eigenvalue of $B_1 \backslash \varepsilon^{-1}K_\varepsilon$, while (2.2) deals with the *first Dirichlet–Neumann* eigenvalue. We doubt the converse implication to be true.

(c) In the special case where $K_\varepsilon = \varepsilon K_1$ it is easy to see that the following equivalences hold: (2.2) $\Leftrightarrow \lambda_1(B_1, K_1) > 0 \Leftrightarrow \lambda_1(\Omega, K_1) > 0 \Leftrightarrow \inf \sigma(H_1) > 0 \Leftrightarrow \inf \sigma_{\text{ess}}(H_1) > 0$.
Furthermore, $\sigma_{\text{ess}}(H_1) > 0$ is equivalent to a Poincaré inequality (2.3) for the space $\mathscr{H}^1(B_1 \backslash K_1)$. This follows from the observation that $\inf \sigma_{\text{ess}}(H_1) > 0$ iff 0 does not belong to the essential spectrum of the Neumann Laplacian of $B_1 \backslash K_1$. The latter equivalence can be established by a simple argument involving singular sequences and cut-offs. Note that, intuitively, singular sequences have to concentrate near ∂K_1. Criteria for the validity of Poincaré inequalities are discussed, e.g., in [8,9,18,19,23].

Examples. In the following examples we restrict our attention to the special case where $K_\varepsilon = \varepsilon K_1$, for simplicity.

(a) If a cone condition holds for $B_1 \backslash K_1$ then the Rellich Compactness Theorem implies that H_1 has compact resolvent and so $\sigma_{\text{ess}}(H_1) = \emptyset$; in particular, Assumption II is satisfied.
(b) Following the construction in [13,14] of *comb*-like domains, one can easily produce examples where $\sigma_{\text{ess}}(H_1) \neq \emptyset$ while $\inf \sigma_{\text{ess}}(H_1) > 0$.
(c) There is a type of *horn* (shrinking at an exponential rate) where the essential spectrum of the Neumann Laplacian begins at $\frac{1}{4}$; cf. [7]. These horns can be wound up to yield a bounded domain, called a "jelly roll" [34]. Here our Assumption II is satisfied.
(d) Further examples are provided by so-called "generalized ridged domains", as discussed in [10,9] and in the literature cited therein.

With $\Omega_\varepsilon = \Omega \backslash K_\varepsilon$ and H_ε as defined in Section 1, for $0 < \varepsilon \leqslant 1$, we are now ready to deal with the essential spectrum of H_ε.

Lemma 2.1. *Let Assumptions* I *and* II *be satisfied. We then have*:

(a) *There exist $c_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\inf \sigma_{\text{ess}}(H_\varepsilon) \geqslant c_0 \, \varepsilon^{-2}, \quad 0 < \varepsilon < \varepsilon_0. \tag{2.4}$$

(b) *If, in addition,* $K_\varepsilon = \varepsilon K_1$, *for* $0 < \varepsilon \leqslant 1$, *then* $\sigma_{\mathrm{ess}}(H_\varepsilon) = \varepsilon^{-2}\sigma_{\mathrm{ess}}(H_1)$, *for* $0 < \varepsilon \leqslant 1$. *In particular,* $\inf \sigma_{\mathrm{ess}}(H_\varepsilon) = \varepsilon^{-2} \inf \sigma_{\mathrm{ess}}(H_1)$.

Proof

(a) Write $\mu_\varepsilon := \inf \sigma_{\mathrm{ess}}(H_\varepsilon)$ and suppose for a contradiction that $\varepsilon^2\mu_\varepsilon \to 0$, as $\varepsilon \to 0$. For each $\varepsilon > 0$, we can find a sequence $(u_n^{(\varepsilon)})_{n\in\mathbb{N}} \subset \mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon)$ satisfying $\|u_n^{(\varepsilon)}\| = 1$, $\|\nabla u_n^{(\varepsilon)}\|^2 \to \mu_\varepsilon$, and $u_n^{(\varepsilon)} \to 0$, weakly in $L_2(\Omega_\varepsilon)$, as $n \to \infty$. Let $\varphi \in C_c^\infty(B_1)$ satisfy $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 1$ for $|x| \leqslant \varrho$, for some fixed $\varrho \in (\zeta, 1)$ where ζ is as in Assumption I. Let $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. Rellich's Compactness Theorem implies that $\|u_n^{(\varepsilon)}\chi_{\{\varrho\varepsilon<|x|<1\}}\| \to 0$, as $n \to \infty$ (after relabelling a subsequence, if necessary) and it follows that $\|\varphi_\varepsilon u_n^{(\varepsilon)}\| \to 1$, as $n \to \infty$. Furthermore, $\|\nabla(\varphi_\varepsilon u_n^{(\varepsilon)})\| \leqslant \|\nabla\varphi_\varepsilon\|_\infty \cdot \|u_n^{(\varepsilon)}\chi_{\{\varrho\varepsilon<|x|<1\}}\| + \sqrt{\mu_\varepsilon} + o(1)$, as $n \to \infty$, and we see that $\limsup_{n\to\infty} \|\nabla(\varphi_\varepsilon u_n^{(\varepsilon)})\|^2 \leqslant \mu_\varepsilon$. Introducing a scaling operator S_ε, acting on functions $g\colon \mathbb{R}^d \to \mathbb{R}$ by $(S_\varepsilon g)(x) := g(\varepsilon x)$, we take the Rayleigh quotient for $S_\varepsilon(\varphi_\varepsilon u_n^{(\varepsilon)})$ to find

$$\lambda_1(B_1, \varepsilon^{-1}K_\varepsilon) \leqslant \liminf_{n\to\infty} \frac{\|\nabla S_\varepsilon(\varphi_\varepsilon u_n^{(\varepsilon)})\|^2}{\|S_\varepsilon(\varphi_\varepsilon u_n^{(\varepsilon)})\|^2} \leqslant \varepsilon^2\mu_\varepsilon \to 0, \quad \varepsilon \to 0, \tag{2.5}$$

in contradiction to (2.2). This concludes the proof.

(b) For $\varepsilon \in (0, 1)$ and $\lambda \in \sigma_{\mathrm{ess}}(H_\varepsilon)$ there exists a singular sequence (or Weyl sequence) $(u_n) \subset D(H_\varepsilon)$ satisfying $u_n \in \mathscr{H}^2_{\mathrm{loc}}(\Omega_\varepsilon)$, $\|u_n\| \leqslant C$, $\liminf \|u_n\| > 0$, $u_n \to 0$ weakly, and $\|(H_\varepsilon - \lambda)u_n\| \to 0$. Let φ and φ_ε be as in part (a) of this proof. We then claim that $(\varphi_\varepsilon u_n)$ is also a singular sequence for H_ε and λ.

To prove this claim, let $w_n := \varphi_\varepsilon u_n$ and observe first that (w_n) is a bounded sequence converging weakly to zero in $L_2(\Omega_\varepsilon)$. Again, it follows from Rellich's Compactness Theorem that there is a subsequence (u_{n_j}) such that $u_{n_j} \to 0$ strongly in $L_2(\Omega\setminus\overline{B}_{\zeta\varepsilon})$ and we see that $\liminf_{j\to\infty} \|\varphi_\varepsilon u_{n_j}\| > 0$. Finally, we have

$$\|(H_\varepsilon - \lambda)w_n\| \leqslant 2\|\nabla\varphi_\varepsilon \cdot \nabla u_n\| + \|\Delta\varphi_\varepsilon u_n\| + \|\varphi_\varepsilon(H_\varepsilon - \lambda)u_n\|. \tag{2.6}$$

Here the second term on the RHS tends to 0 (at least for the subsequence (u_{n_j})) as $u_{n_j} \to 0$ strongly in $L_2(B_1\setminus\overline{B}_{\varepsilon\varrho})$. To estimate the first term on the RHS, we use the well-known identity [33, Lemma C.2.1]

$$|\nabla f|^2 = \tfrac{1}{2}\Delta|f|^2 - \operatorname{Re}(\overline{f}\Delta f), \quad f \in \mathscr{H}^2_{\mathrm{loc}}, \tag{2.7}$$

which gives

$$\|\nabla\varphi_\varepsilon \cdot \nabla u_n\|^2 \leqslant \||\nabla\varphi_\varepsilon|\,|\nabla u_n|\|^2 \leqslant \frac{1}{2}\int_{\Omega_\varepsilon} |u_n|^2\Delta|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} |u_n||\Delta u_n||\nabla\varphi_\varepsilon|^2\,\mathrm{d}x, \tag{2.8}$$

and the claim follows.

To conclude the proof, we define $W_n \in D(H_1)$ by $W_n(x) = w_n(\varepsilon x)$ and find that (W_n) is a singular sequence for H_1 and $\varepsilon^2\lambda$. Hence $\sigma_{\mathrm{ess}}(H_1) \supset \varepsilon^2\sigma_{\mathrm{ess}}(H_\varepsilon)$. The proof for the converse direction is similar and omitted. □

In the following lemma, a simple upper bound for the first eigenvalue $\lambda_1(\varepsilon)$ is obtained by variational methods. Note that the statement of Lemma 2.2 allows for $|K_\varepsilon| = 0$.

Lemma 2.2. *Suppose Assumptions* I *and* II *are satisfied and let* H_ε *and* $\lambda_1(\varepsilon)$ *be as above. Then there exist real constants* c *and* c' *such that*

$$\lambda_1(\varepsilon) \leqslant \Lambda_1 + c|K_\varepsilon| \leqslant \Lambda_1 + c'\varepsilon^d, \quad 0 < \varepsilon \leqslant 1. \tag{2.9}$$

Proof. Let $U_1 \in D(H^\Omega) \subset \mathscr{H}^1_\circ(\Omega)$ denote the normalized ground state eigenfunction for the Dirichlet problem on Ω, i.e., $H^\Omega U_1 = \Lambda_1 U_1$, and let $v_\varepsilon = \chi_{\Omega_\varepsilon} U_1$. Trivially, $\|\nabla v_\varepsilon\|^2 \leqslant \|\nabla U_1\|^2 = \Lambda_1$. Since U_1 is a bounded function we have $\|v_\varepsilon\|^2 \geqslant 1 - c_0|K_\varepsilon|$ and it follows that

$$\lambda_1(\varepsilon) \leqslant \|\nabla v_\varepsilon\|^2 \cdot \|v_\varepsilon\|^{-2} \leqslant \Lambda_1(1 - c_0\,|K_\varepsilon|)^{-1}, \quad 0 < \varepsilon \leqslant \varepsilon_0, \tag{2.10}$$

with $\varepsilon_0 > 0$ so small that $c_0|B_{\varepsilon_0}| \leqslant 1/2$. It is easy to modify the above argument to obtain an upper bound $\lambda_1(\varepsilon) \leqslant C$, valid for $0 < \varepsilon \leqslant 1$, and we are done. □

Remark. The question whether $\lambda_1(\varepsilon)$ is smaller or larger than Λ_1 is left open in Lemma 2.2. The variational argument used in the above proof can be refined to show that the constant c in (2.9) is negative if 0 is close to the boundary of Ω (note that $U_1(x)$ is small near $\partial\Omega$ while ∇U_1 is of order 1). Conversely, one would expect that $\lambda_1(\varepsilon) > \Lambda_1$ for small $\varepsilon > 0$ if U_1 happens to have a relative maximum at 0. These variational heuristics are in agreement with Ozawa's asymptotic formula [25], valid in $\mathbb{R}^2$ for $K_\varepsilon = \overline{B}_\varepsilon$,

$$\lambda_1(\varepsilon) = \Lambda_1 - (2\pi|\nabla U_1(0)|^2 - \pi\Lambda_1|U_1(0)|^2)\varepsilon^2 + O(\varepsilon^3|\log\varepsilon|^2), \quad \varepsilon \to 0, \tag{2.11}$$

and with Hadamard's formula [25,26] for the derivative $\lambda_1'(\varepsilon)$; cf. also [21, vol. I, p. 318].

In the following lemma, we draw a consequence from Assumption II for scaled versions of the compact sets K_ε. The estimate (2.12), given below, could be considered as a preliminary asymptotic lower bound for the eigenvalues for rescaled Neumann obstacles.

Lemma 2.3. *Let Assumptions* I *and* II *be satisfied and let* $\lambda_1(B_1, \eta^{-1}K_\varepsilon)$ *denote the first min–max value of the Laplacian on* $B_1 \backslash \eta^{-1}K_\varepsilon$ *with Dirichlet boundary condition on* ∂B_1 *and Neumann boundary condition on* $\eta^{-1}\partial K_\varepsilon$, *for* $0 < \varepsilon \leqslant \eta \leqslant 1$. *Then there is a constant* $c_0 > 0$ *such that*

$$\liminf_{\eta\to 0}\ \liminf_{\varepsilon\to 0}\ \lambda_1(B_1, \eta^{-1}K_\varepsilon) \geqslant c_0 > 0. \tag{2.12}$$

Proof. Suppose (2.12) is not true. Then we can find sequences $\eta_n \to 0$ and $\varepsilon_n \to 0$ such that $0 < \varepsilon_n \leqslant \eta_n$ and

$$\alpha_n := \lambda_1(B_1, \eta_n^{-1}K_{\varepsilon_n}) \to 0, \quad n \to \infty. \tag{2.13}$$

Let $v_n \in \mathscr{H}^1_{\mathrm{DN}}(B_1 \backslash \eta_n^{-1}K_{\varepsilon_n})$ satisfy $\|v_n\| = 1$, $\|\nabla v_n\|^2 \leqslant 2\alpha_n$, and choose a function $\varphi \in C_c^\infty(B_1)$ satisfying $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 1$ for $|x| \leqslant \zeta$. Let $\varphi_\kappa(x) := \varphi(\kappa^{-1}x)$, for $\kappa > 0$, and let $\Phi_n = \varphi_{\varepsilon_n/\eta_n}$. By Lemma 5.4 and (2.13) we then find that

$$\|\chi_{\{\zeta\varepsilon_n/\eta_n < |x| < 1\}}\, v_n\|^2 \leqslant c_1\|\nabla v_n\|^2 \leqslant 2c_1\alpha_n \to 0, \tag{2.14}$$

in particular, $\|\Phi_n v_n\| \geqslant 1 - \sqrt{2c_1\alpha_n} \to 1$, as $n \to \infty$. Furthermore,

$$\begin{aligned}\|\nabla(\Phi_n v_n)\|^2 &\leqslant 4\alpha_n + c_2(\eta_n/\varepsilon_n)^2 \, \|\chi_{\{\zeta\varepsilon_n/\eta_n<|x|<1\}} \, v_n\|^2 \\ &\leqslant c_3\alpha_n(1+(\eta_n/\varepsilon_n)^2).\end{aligned} \tag{2.15}$$

Taking the Rayleigh quotient for $\Phi_n v_n$ we obtain from the above

$$\lambda_1(B_{\varepsilon_n/\eta_n}, \eta_n^{-1}K_{\varepsilon_n}) \leqslant \frac{\|\nabla(\Phi_n v_n)\|^2}{\|\Phi_n v_n\|^2} \leqslant 2c_3\alpha_n(1+(\eta_n/\varepsilon_n)^2), \tag{2.16}$$

for n large. Scaling out by a factor of η_n/ε_n (for n large), we finally get

$$\lambda_1(B_1, \varepsilon_n^{-1}K_{\varepsilon_n}) \leqslant c_4(\varepsilon_n/\eta_n)^2\alpha_n(1+(\eta_n/\varepsilon_n)^2) \to 0, \quad n \to \infty, \tag{2.17}$$

in contradiction to (2.2). $\square$

The following theorem is our first main result. Here we show under rather weak assumptions that the first eigenvalue $\lambda_1(\varepsilon)$ of H_ε converges to the corresponding Dirichlet eigenvalue Λ_1 of Ω. We denote as $H = H^\Omega$ the Dirichlet Laplacian of Ω, with eigenvalues Λ_j and an associated orthonormal basis of eigenfunctions $(U_j)_{j\in\mathbb{N}}$, as in Section 1.

Theorem 2.4. *Let Ω and $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ satisfy Assumptions* I *and* II. *We then have*:

(a) *There exists $\varepsilon_0 > 0$ such that $\lambda_1(\varepsilon)$ is a discrete eigenvalue of H_ε, for $0 < \varepsilon \leqslant \varepsilon_0$.*
(b) $\lambda_1(\varepsilon) \to \Lambda_1$, *as* $\varepsilon \to 0$.
(c) *Let u_ε denote a normalized eigenfunction for H_ε and $\lambda_1(\varepsilon)$, extended by* 0 *to K_ε. Then $u_\varepsilon \to U_1$ strongly in $L_2(\Omega)$, as $\varepsilon \to 0$.*

Proof. Part (a) is immediate from Lemmas 2.1 and 2.2.

Let $\varepsilon_0 > 0$ be as in (a) and let $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, \varepsilon_0)$ with $\varepsilon_n \to 0$. By Lemma 2.2, the eigenvalues $\lambda_1(\varepsilon_n)$ have an accumulation point $\tilde{\lambda} \in [0, \Lambda_1]$. We may assume that $\lambda_1(\varepsilon_n) \to \tilde{\lambda}$, as $n \to \infty$.

(1) There is a constant c_0 such that $\int_{\Omega\setminus B_\eta} |\nabla u_\varepsilon|^2 \, dx \leqslant c_0$, for $0 < \varepsilon < \eta \leqslant \varepsilon_0$. Routine arguments (involving the repeated selection of appropriate subsequences which we do not make explicit in the notation) yield that there exists a function $u_0 \in L_2(\Omega)$ such that $\|u_0\| \leqslant 1$ and, as $n \to \infty$,

$$\begin{aligned} &u_{\varepsilon_n} \to u_0, \quad \text{weakly in } L_2(\Omega), \\ &u_{\varepsilon_n} \to u_0, \quad \text{strongly in } L_2(\Omega\setminus B_\eta), \text{ for any } \eta > 0, \\ &u_{\varepsilon_n} \to u_0, \quad \text{pointwise almost everywhere.}\end{aligned} \tag{2.18}$$

Furthermore, for any $\eta > 0$, we have $u_0 \in \mathscr{H}^1_{\mathrm{DN}}(\Omega\setminus B_\eta)$ and $\nabla u_{\varepsilon_n} \to \nabla u_0$, weakly in $L_2(\Omega\setminus B_\eta)^d$. It follows that $\int_{\Omega\setminus B_\eta} |\nabla u_0|^2 \, dx \leqslant c_0$, for all $\eta > 0$, and thus $\int_{\Omega\setminus\{0\}} |\nabla u_0|^2 \, dx < \infty$. We therefore see that $u_0 \in \mathscr{H}^1_{\mathrm{DN}}(\Omega\setminus\{0\}) = \mathscr{H}^1_\circ(\Omega)$, by (1.9). Also, u_0 satisfies

$$\langle \nabla u_0 \, , \, \nabla\varphi \rangle = \tilde{\lambda}\langle u_0 \, , \, \varphi\rangle, \quad \varphi \in C_c^\infty(\Omega\setminus\{0\}); \tag{2.19}$$

by Lemma 1.1, (2.19) extends to all $\varphi \in C_c^\infty(\Omega)$. As a consequence, $\tilde{\lambda}$ agrees with one of the eigenvalues Λ_k of the Dirichlet Laplacian H, unless $u_0 = 0$. Since we have seen above that $\tilde{\lambda} \leqslant \Lambda_1$, we conclude that $\tilde{\lambda} = \Lambda_1$ and $u_0 = U_1$, *provided* $\|u_0\| = 1$.

(2) We now show $\|u_0\| = 1$. Let us assume, for a contradiction, that $s := \|u_0\| < 1$. Let $\varphi \in C_c^\infty(B_1)$ satisfy $\varphi(x) = 1$ for $|x| \leqslant 1/2$, $0 \leqslant \varphi \leqslant 1$, and define $\varphi_\eta(x) = \varphi(\eta^{-1}x)$. Writing $v_{\varepsilon_n} = u_{\varepsilon_n} - u_0$ we have $\|v_{\varepsilon_n}\| \geqslant 1 - s$ and $(1 - \varphi_\eta)v_{\varepsilon_n} \to 0$ in L_2, as $n \to \infty$, by (2.18), whence $\|\varphi_\eta v_{\varepsilon_n}\| \geqslant 1 - s + o(1)$, as $n \to \infty$. Furthermore, $\|\nabla(\varphi_\eta v_{\varepsilon_n})\| \leqslant c_1\eta^{-1}\|v_{\varepsilon_n}\chi_{\{\eta/2<|x|<\eta\}}\| + c_2$, with constants c_1, c_2 independent of ε_n and η. Now min–max combined with $v_{\varepsilon_n}\chi_{\{\eta/2<|x|<\eta\}} \to 0$ in L_2, as $n \to \infty$, implies that

$$\lambda_1(B_\eta, K_{\varepsilon_n}) \leqslant \frac{\|\nabla(\varphi_\eta v_{\varepsilon_n})\|^2}{\|\varphi_\eta v_{\varepsilon_n}\|^2} \leqslant \frac{c_2^2}{(1-s)^2} + \mathrm{o}(1), \quad n \to \infty, \tag{2.20}$$

for any fixed $\eta > 0$. Scaling out by a factor of η we find

$$\limsup_{n\to\infty} \lambda_1(B_1, \eta^{-1}K_{\varepsilon_n}) \leqslant c_3\eta^2, \quad 0 < \eta < 1, \tag{2.21}$$

with some constant c_3 that is independent of η. On the other hand, it follows from Lemma 2.3 that

$$\eta^{-2} \liminf_{\varepsilon\to 0} \lambda_1(B_1, \eta^{-1}K_\varepsilon) \to \infty, \quad \eta \to 0, \tag{2.22}$$

which gives the desired contradiction. □

Remarks

(a) It follows from Theorem 2.4 that $\lambda_1(\varepsilon)$ is simple for $\varepsilon > 0$ small. We expect, more generally, that the ground state for $-\Delta$ on a bounded, connected domain $G \subset \mathbb{R}^d$ with mixed Dirichlet–Neumann boundary conditions is always simple.
(b) As for the higher eigenvalues $\lambda_k(\varepsilon)$, $k \geqslant 2$, the above proofs can be modified to yield convergence $\lambda_k(\varepsilon) \to \Lambda_k$, as $\varepsilon \to 0$, for all $k \in \mathbb{N}$.
(c) If $K_\varepsilon = \varepsilon K_1$, then λ_1 is in fact continuous on the closed interval $[0, 1]$. While continuity at $\varepsilon = 0$ follows from Theorem 2.4, continuity at $\varepsilon \in (0, 1]$ can be shown by simple, but rather lengthy, min–max arguments of which we only give a sketch here.

For simplicity, let us consider the case where $\sigma_{\mathrm{ess}}(H_1) = \emptyset$. Let ζ be as in Assumption I so that $K_1 \subset B_\zeta$ and choose a function $\varphi \in C_c^\infty(B_1)$ satisfying $0 \leqslant \varphi \leqslant 1$ and $\varphi\restriction_{B_\zeta} = 1$. We wish to compare the eigenvalues $\lambda_1(\varepsilon)$ and $\lambda_1(\varepsilon')$, for $\varepsilon - \varepsilon'$ small, where we consider $\varepsilon \in (0, 1]$ fixed. We scale the eigenfunction u_ε by a factor of ε/ε' and use the cut-off φ to glue together the scaled part inside B_1 with the unmodified part in $\Omega \backslash B_1$, i.e., we consider

$$u'(x) := \varphi(x)u_\varepsilon(\alpha x) + (1 - \varphi)u_\varepsilon(x), \tag{2.23}$$

where $\alpha := \varepsilon/\varepsilon'$, and note that $u' \in \mathscr{H}^1_{\mathrm{DN}}(\Omega_{\varepsilon'})$. By elliptic regularity theory, we may assume that u_ε is uniformly continuous with uniformly continuous first derivatives on $\overline{B}_1 \backslash B_\zeta$, with universal bounds on the function and its first derivatives. It follows that $\lambda_1(\varepsilon') \leqslant \|\nabla u'\|^2/\|u'\|^2$, with $\|\nabla u'\|^2 \to \|\nabla u_\varepsilon\|^2$ and $\|u'\|^2 \to 1$, as $\varepsilon' \to \varepsilon$, as one can show without great difficulty. Hence $\limsup_{\varepsilon'\to\varepsilon} \lambda_1(\varepsilon') \leqslant \lambda_1(\varepsilon)$. The proof for the converse estimate $\liminf_{\varepsilon'\to\varepsilon} \lambda_1(\varepsilon') \geqslant \lambda_1(\varepsilon)$ is similar.

To conclude this section, we prove that condition (2.1) implies condition (2.3).

Suppose for a contradiction that (2.1) holds while (2.3) is not satisfied. Then there are sequences $\varepsilon_n \to 0$ and $(u_n) \subset \mathscr{H}^1(B_1 \backslash \varepsilon_n^{-1} K_{\varepsilon_n})$ such that

$$\|u_n - \overline{u}_n\| = 1, \quad \|\nabla u_n\| \to 0. \tag{2.24}$$

Let $\varphi \in C_c^\infty(B_1)$ satisfy $0 \leqslant \varphi \leqslant 1$ and $\varphi\restriction_{B_{\varrho'}} = 1$ where $\zeta < \varrho' < 1$. Also write $\psi = 1 - \varphi$, $M = B_1 \backslash B_{\varrho'}$ and $m_n = \overline{u_n\restriction_M}$, the mean of u_n over M. By (2.24) and a Poincaré inequality for the region M, we have $\|(u_n - m_n)\restriction_M\| \to 0$, as $n \to \infty$. We now get

$$\begin{aligned} \|u_n - \overline{u}_n\| &\leqslant \|u_n - m_n\| \leqslant \|\varphi(u_n - m_n)\| + \|\psi(u_n - m_n)\| \\ &\leqslant c_1\|\nabla(\varphi(u_n - m_n))\| + c_2\|\nabla(\psi(u_n - m_n))\| \\ &\leqslant c_3\|\nabla u_n\| + c_4\|\nabla\varphi\|_\infty \|(u_n - m_n)\restriction_M\| \to 0, \quad n \to \infty, \end{aligned} \tag{2.25}$$

in the third step we have used (2.2), which is equivalent to (2.1), to estimate $\|\varphi(u_n - c_n)\|$, while $\|\psi(u_n - c_n)\|$ can be estimated by a Poincaré-type inequality for the "annulus" $B_1 \backslash B_{\varrho'}$ with Dirichlet boundary condition at the inner radius and Neumann boundary condition at the outer radius. This gives the desired contradiction.

3. The case $K_\epsilon = \overline{B}_\epsilon$

In this section, we discuss the special case $K_\varepsilon = \overline{B}_\varepsilon$ where we provide the estimate

$$|\lambda_1(\varepsilon) - \Lambda_1| \leqslant c\,\varepsilon^d, \quad \varepsilon \to 0. \tag{3.1}$$

This estimate will be a crucial ingredient in the next section when we attack general obstacles. For $d=2$, 3, (3.1) follows directly from the more precise asymptotic estimates of [21, Chapter 9], or from Ozawa's estimate (2.11). We include a simple, independent proof of (3.1) based on an expansion in terms of Bessel functions.

The upper bound $\lambda_1 \leqslant \Lambda_1 + c\varepsilon^d$, for $0 < \varepsilon < \varepsilon_0$, is covered by Lemma 2.2. For the lower bound, we use extension of the eigenfunctions u_ε into B_ε. This extension process must satisfy two requirements: the extended function has to be in $\mathscr{H}^1_\circ(\Omega)$, and we need an estimate for the gradient norm of the extensions on B_ε. Following [28,2], the harmonic extension is a natural choice. To keep notation simple we restrict ourselves to the case $d = 2$ in the sequel. It is an easy, but somewhat lengthy, exercise to provide corresponding results for $d > 2$, using the relevant special functions [22].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain containing $\overline{B}_1$. For $0 < \varepsilon \leqslant 1$ let $\lambda_1(\varepsilon) = \lambda_1(\Omega, \overline{B}_\varepsilon)$ with associated normalized eigenfunctions u_ε. Let $w_\varepsilon : \overline{B}_\varepsilon \to \mathbb{R}$ denote the harmonic extension of u_ε to B_ε, i.e., w_ε is harmonic in B_ε, continuous on $\overline{B}_\varepsilon$ and satisfies $w_\varepsilon(x) = u_\varepsilon(x)$ for $|x| = \varepsilon$.*

Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that $\|\nabla w_\varepsilon\|^2 \leqslant C\varepsilon^2$, for $0 < \varepsilon < \varepsilon_0$.

Proof. (1) Since $\lambda_1(\varepsilon) \to \Lambda_1$, by Theorem 2.4, we can find $\varepsilon_1 > 0$ such that $\lambda_1(\varepsilon) \in [\Lambda_1/2, 2\Lambda_1]$, for $0 < \varepsilon \leqslant \varepsilon_1$. Furthermore, there exists $\delta_1 > 0$ such that the Bessel functions of the second kind, Y_n, satisfy $Y_n'(\mu r) \neq 0$ for all $n \in \mathbb{N}$, $0 < \mu \leqslant \sqrt{2\Lambda_1}$, and $0 < r < \delta_1$; cf. (5.4). We now fix $\delta \in (0, \min\{1, \delta_1\})$.

As $\|\nabla u_\varepsilon\|^2 \leqslant c_1$, $0 < \varepsilon \leqslant 1$, it follows by the Sobolev Trace Theorem that the family $(u_\varepsilon\restriction_{\partial B_\delta})_{0<\varepsilon\leqslant\delta/2}$ is bounded in $L_2(\partial B_\delta)$. Since u_ε is a smooth function (up to the boundary of B_ε) we may expand u_ε for

$|x|=\delta$ and for $|x|=\varepsilon$ in the form $u_\varepsilon(x)=\sum a_{n;\varepsilon}\mathrm{e}^{\mathrm{i}n\vartheta}$, for $x=\delta\mathrm{e}^{\mathrm{i}\vartheta}$, and $u_\varepsilon(x)=\sum b_{n;\varepsilon}\mathrm{e}^{\mathrm{i}n\vartheta}$, for $x=\varepsilon\mathrm{e}^{\mathrm{i}\vartheta}$, with (a_n) and (b_n) square summable. The uniform bound on $u_\varepsilon\restriction_{\partial B_\delta}$ in the norm of $L_2(\partial B_\delta)$ translates into

$$\sum |a_{n;\varepsilon}|^2 \leqslant c_2, \quad 0<\varepsilon\leqslant\delta/2. \tag{3.2}$$

Below, it will be shown that

$$|b_{n;\varepsilon}| \leqslant c_3 \frac{\varepsilon^n}{\delta^n}|a_{n;\varepsilon}|, \quad n\in\mathbb{Z}, \; 0<\varepsilon\leqslant\delta/2. \tag{3.3}$$

Now, the harmonic function w_ε in B_ε with boundary values $u\restriction\partial B_\varepsilon$ is given by

$$w_\varepsilon(x)=\sum_{n\in\mathbb{Z}}(r/\varepsilon)^n b_{n;\varepsilon}\mathrm{e}^{\mathrm{i}n\vartheta}, \quad 0\leqslant r\leqslant\varepsilon, \; x=r\mathrm{e}^{\mathrm{i}\vartheta}. \tag{3.4}$$

By a well-known formula [16, p. 125] and the fact that the gradient norm is invariant under scaling in $\mathbb{R}^2$, the gradient norm of w_ε satisfies

$$\|\nabla w_\varepsilon\|^2=\pi\sum_{n\neq 0} n|b_{n;\varepsilon}|^2 \leqslant c_4\sum_{n\neq 0} n\frac{\varepsilon^{2n}}{\delta^{2n}}|a_{n;\varepsilon}|^2 \leqslant c_5\varepsilon^2, \tag{3.5}$$

as claimed, where we have used (3.3) in the last step.

(2) It remains to prove (3.3). Here we use an obvious expansion of u_ε in the annulus $B_\delta\backslash B_\varepsilon$ connecting the expansion of u_ε at $|x|=\delta$ to the expansion at $|x|=\varepsilon$:

Let $\mu=\mu(\varepsilon)=\sqrt{\lambda_1(\varepsilon)}$ where, as above, $\lambda_1(\varepsilon)\in[\Lambda_1/2, 2\Lambda_1]$, for $0<\varepsilon<\varepsilon_1$. For each $n\in\mathbb{N}_0$, let $J_n=J_n(r)$ denote the Bessel function of the first kind. Then u_ε has an expansion

$$u_\varepsilon(x)=\sum_{n\in\mathbb{N}_0} a_{n;\varepsilon}\Phi_{n;\varepsilon}(r)\mathrm{e}^{\mathrm{i}n\vartheta}, \quad \varepsilon\leqslant r\leqslant\delta, \; x=r\mathrm{e}^{\mathrm{i}\vartheta}, \tag{3.6}$$

where each function $\Phi_{n;\varepsilon}$ is a linear combination of $J_n(\mu(\varepsilon)r)$ and $Y_n(\mu(\varepsilon)r)$, chosen in such a way that the boundary conditions at $|x|=\varepsilon$ and at $|x|=\delta$ are satisfied, i.e., $\Phi'_{n;\varepsilon}(\varepsilon)=0$ and $\Phi_{n;\varepsilon}(\delta)=1$. The estimate (3.3) now follows from $b_{n;\varepsilon}=a_{n;\varepsilon}\Phi_{n;\varepsilon}(\varepsilon)$ and Lemma 5.3. □

The following theorem states the basic estimate for the case $K_\varepsilon=\overline{B}_\varepsilon$.

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with $\overline{B}_1\subset\Omega$ and let $K_\varepsilon=\overline{B}_\varepsilon$. Then there is a constant $c\geqslant 0$ and $\varepsilon_0>0$ such that*

$$|\lambda_1(\varepsilon)-\Lambda_1|\leqslant c\varepsilon^d, \quad 0<\varepsilon<\varepsilon_0. \tag{3.7}$$

Proof. (Only given for $d=2$.) We only need to provide a lower bound for $\lambda_1(\varepsilon)$. Let w_ε: $\overline{B}_\varepsilon\to\mathbb{R}$ denote the harmonic extension of u_ε as in Lemma 3.1, and let $\tilde{u}_\varepsilon$ agree with u_ε on $\Omega_\varepsilon=\Omega\backslash\overline{B}_\varepsilon$ and with w_ε in B_ε. Then $w_\varepsilon\in\mathscr{H}^1_\circ(\Omega)$, $\|w_\varepsilon\|\geqslant\|u_\varepsilon\|=1$ and, by Lemma 3.1, $\|\nabla w_\varepsilon\|^2\leqslant\|\nabla u_\varepsilon\|^2+C\varepsilon^2$, for $0<\varepsilon<\varepsilon_0$. Now the Rayleigh–Ritz variational formula implies $\Lambda_1\leqslant\|\nabla w_\varepsilon\|^2/\|w_\varepsilon\|^2\leqslant\lambda_1(\varepsilon)+C\varepsilon^2$, and we are done. □

The result of Theorem 3.2 extends to all eigenvalues $\lambda_k(\varepsilon)$ but the proof requires some more work. Specifically, we would also need to control the L_2-norm of the harmonic extensions in B_ε via the maximum principle. This in turn requires that the restriction of the eigenfunctions (associated to a bounded sequence of eigenvalues) to ∂B_ε is uniformly bounded. A bound of this type follows from (3.2–4) by inspection.

4. General obstacles

We finally implement the idea of [3] of interpolating a smooth "comparison problem" to obtain quantitative lower bounds on $\lambda_1(\varepsilon)$ that depend on a priori estimates on u_ε in L_p-norm, for some $p > 2$. Here the comparison problem will be the one studied in Section 3 with obstacles given by a ball of radius ε. We present the basic variational argument in Proposition 4.1. The main difficulty then lies in obtaining suitable L_p-bounds from regularity assumptions on ∂K_ε. Here we first study K_ε with Lipshitz boundary where we get estimates for any $p < \infty$ in $\mathbb{R}^2$ and for $p = \frac{2d}{d-2}$ in $\mathbb{R}^d$, $d > 2$, following [28]. Finally, we present a case where an a priori estimate in the maximum-norm is possible.

Proposition 4.1. *Let Ω and $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ satisfy Assumptions* I *and* II *and let u_ε and ε_0 be as in Theorem* 2.4. *In addition, suppose we have an a priori estimate*

$$\|u_\varepsilon\restriction_{B_\varepsilon\setminus K_\varepsilon}\|_p \leqslant C, \quad 0 < \varepsilon < \varepsilon_0, \tag{4.1}$$

for some $p \in (2, \infty]$. Then there exists a constant $c \geqslant 0$ such that

$$\lambda_1(\varepsilon) \geqslant \Lambda_1 - c\varepsilon^{d-2d/p}, \quad 0 < \varepsilon < \varepsilon_0; \tag{4.2}$$

for $p = \infty$ and $r \neq 0$, we read $r/p = 0$.

Proof. From the assumption we deduce

$$\|u_\varepsilon\restriction_{B_\varepsilon\setminus K_\varepsilon}\|_2^2 \leqslant c_1 |B_\varepsilon|^{1-2/p} = c_2\varepsilon^{d-2d/p}. \tag{4.3}$$

Upon insertion of a Neumann boundary condition along ∂B_ε we obtain the Sobolev spaces

$$\mathscr{M}_\varepsilon = \mathscr{H}^1(B_\varepsilon\setminus K_\varepsilon) \oplus \mathscr{H}^1_{\mathrm{DN}}(\Omega\setminus\overline{B}_\varepsilon) \supset \mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon) \tag{4.4}$$

and find

$$\begin{aligned} \lambda_1(\varepsilon) &= \|\nabla u_\varepsilon\|^2 \\ &\geqslant \inf\{\|\nabla u\|^2;\ u \in \mathscr{M}_\varepsilon,\ \|u\| = 1,\ \|u\restriction_{B_\varepsilon\setminus K_\varepsilon}\|^2 \leqslant c_2\varepsilon^{d-2d/p}\} \\ &\geqslant \inf\{\|\nabla v\|^2;\ v \in \mathscr{H}^1_{\mathrm{DN}}(\Omega\setminus\overline{B}_\varepsilon),\ \|v\|^2 \geqslant 1 - c_2\varepsilon^{d-2d/p}\} \\ &\geqslant \tilde{\lambda}_1(\varepsilon)(1 - c_2\varepsilon^{d-2d/p}), \end{aligned} \tag{4.5}$$

where $\tilde{\lambda}_1(\varepsilon)$ denotes the first DN eigenvalue of $\Omega\setminus\overline{B}_\varepsilon$. By Theorem 3.2, we have $\tilde{\lambda}_1(\varepsilon) \geqslant \Lambda_1 - c_3\varepsilon^d$, and thus

$$\lambda_1(\varepsilon) \geqslant (\Lambda_1 - c_3\varepsilon^d)(1 - c_2\varepsilon^{d-2d/p}) \geqslant \Lambda_1 - c_4\varepsilon^{d-2d/p}. \qquad \square \tag{4.6}$$

Remark. Some additional effort would be necessary to extend Proposition 4.1 to the higher eigenvalues $\lambda_k(\varepsilon)$, $k \geqslant 2$, since we would also need to control the rate of convergence of the associated eigenfunctions.

For simplicity of the presentation, we will restrict our attention to the special case $K_\varepsilon = \varepsilon K_1$ for the remainder of this section. An easy way to obtain a priori estimates on u_ε is via elliptic regularity theory. In Corollary 4.2, below, we follow [28, Example 1, p. 40] to obtain uniform control of the extension operator. Here we need to assume that ∂K_1 is Lipshitz. Since Lipshitz regularity implies a cone condition, Assumption II is clearly satisfied.

Corollary 4.2. *Let Assumption* I *be satisfied and suppose*, *in addition*, *that* $K_\varepsilon = \varepsilon K_1$ *with* ∂K_1 *Lipshitz.*

(a) *Let* $d = 2$. *Then for any* $\gamma > 0$ *there exist* $c_\gamma \geqslant 0$ *and* $\varepsilon_\gamma > 0$ *such that*

$$\lambda_1(\varepsilon) \geqslant \Lambda_1 - c_\gamma \varepsilon^{2-\gamma}, \quad 0 < \varepsilon < \varepsilon_\gamma. \tag{4.7}$$

(b) *Let* $d \in \mathbb{N}$, $d > 2$. *Then there exist* c_0 *and* $\varepsilon_0 > 0$ *such that*

$$\lambda_1(\varepsilon) \geqslant \Lambda_1 - c_0 \varepsilon^2, \quad 0 < \varepsilon < \varepsilon_0. \tag{4.8}$$

Proof. As ∂K_1 is Lipshitz, there is a continuous Sobolev extension operator

$$E : \mathscr{H}^1(B_1 \backslash K_1) \to \mathscr{H}^1(B_1). \tag{4.9}$$

For $u \in \mathscr{H}^1(B_1 \backslash K_1)$ write $\tilde{u} = Eu$. We then have

$$\|\tilde{u}\|_{L_2(B_1)} \leqslant C_1 \|u\|_{\mathscr{H}^1(B_1 \backslash K_1)}, \quad \|\nabla \tilde{u}\|_{L_2(B_1)} \leqslant C_2 \|\nabla u\|_{L_2(B_1 \backslash K_1)}; \tag{4.10}$$

the first inequality is immediate while the second one follows from

$$\|\nabla \tilde{u}\|_{L_2(B_1)} \leqslant C_3 \|u\|_{\mathscr{H}^1(B_1 \backslash K_1)} \tag{4.11}$$

and the Poincaré inequality (2.3) much as in [28, p. 40]. Scaling the inequalities (4.10) yields a family of extension operators

$$E_\varepsilon : \mathscr{H}^1(B_\varepsilon \backslash K_\varepsilon) \to \mathscr{H}^1(B_\varepsilon), \tag{4.12}$$

with uniformly bounded norm, $\|E_\varepsilon\| \leqslant c$.

For a proof of (a), let $v_\varepsilon = u_\varepsilon \restriction_{B_\varepsilon \backslash K_\varepsilon}$, where u_ε is again a normalized eigenfunction of H_ε associated with the eigenvalue $\lambda_1(\varepsilon)$. Defining $\tilde{u}_\varepsilon = u_\varepsilon$ on Ω_ε, and $\tilde{u}_\varepsilon = E_\varepsilon v_\varepsilon \restriction_{K_\varepsilon}$ on K_ε we have $\|\tilde{u}_\varepsilon\|_{\mathscr{H}^1(\Omega)} \leqslant c$ and the Sobolev Embedding Theorem implies that

$$\|\tilde{u}_\varepsilon \restriction_{B_1}\|_p \leqslant c_p, \quad 1 \leqslant p < \infty. \tag{4.13}$$

The desired result is now immediate from Proposition 4.1. The proof of part (b) is analogous and omitted. $\square$

Remark. Of course, the upper estimate $\lambda_1(\varepsilon) \leqslant \Lambda_1 + c\varepsilon^d$, for $0 < \varepsilon < \varepsilon_1$, from Lemma 2.3 holds under the assumptions of Corollary 4.2.

We finally discuss a simple situation where a uniform bound on the maximum of the eigenfunctions u_ε can be obtained. We assume, roughly speaking, that K_1 is a limit of star-shaped sets that approximate K_1 from the outside.

Assumption III. $0 \in K_1$ and there is a sequence of open sets $G_n \subset B_1$ such that $\overline{G}_n \subset B_1$ and

(1) $K_1 \subset G_n \subset G_{n-1}$, for $n \in \mathbb{N}$, and $|K_1 - G_n| \to 0$ as $n \to \infty$,
(2) ∂G_n is smooth and at each $x \in \partial G_n$ the interior normal vector $v(x)$ satisfies $\langle x\,,\ v(x)\rangle \leqslant 0$.

Examples. K_1 may be a line segment, or K_1 may possess radial "spikes" pointing outwards. K_1 may also have cusps; no cone condition is required for $B_1 \backslash K_1$.

Lemma 4.3. *Suppose Ω and K_ε satisfy Assumptions* I–III *with $K_\varepsilon = \varepsilon K_1$, and let u_ε be as above. Then there exist $\delta > 0$, $\varepsilon_0 \in (0, \delta)$, and a constant M such that for $0 < \varepsilon \leqslant \varepsilon_0$ we have*

$$0 \leqslant u_\varepsilon(x) \leqslant M, \quad x \in B_\delta \backslash K_\varepsilon. \tag{4.14}$$

Proof. The following proof is given for $d = 2$ but generalizes easily to higher dimensions by replacing the Bessel function J_0 with the corresponding special function.

(1) By Lemma 2.2 there is a constant C_1 and an $\varepsilon_1 > 0$ such that $\lambda_1(\varepsilon) \leqslant C_1$, for $0 < \varepsilon \leqslant \varepsilon_1$. Furthermore, Assumption II and scaling imply that there exists a constant $c_1 > 0$ such that $\|\nabla u\|^2 \geqslant c_1 \delta^{-2} \|u\|^{-2}$, for all $u \in \mathscr{H}^1(B_\delta \backslash K_\varepsilon)$ that vanish at $|x| = \delta$; here $\varepsilon \in (0, \delta)$. We now fix $\delta = \delta_0 > 0$ so that $c_1/\delta_0^2 > C_1$.

(2) As u_ε converges to U_1 in $L_2(\Omega \backslash B_\varrho)$, for any $\varrho > 0$, elliptic regularity theory implies that u_ε converges to U_1 uniformly on compact subsets of $\Omega \backslash \{0\}$, as $\varepsilon \to 0$; in particular, there is a constant C_2 such that $0 \leqslant u_\varepsilon(x) \leqslant C_2$, for $|x| = \delta_0$ and $0 < \varepsilon \leqslant \varepsilon_0$. Define comparison functions

$$v_\varepsilon(x) := \alpha_\varepsilon J_0(\sqrt{\lambda_1(\varepsilon)}|x|), \tag{4.15}$$

with a constant α_ε chosen in such a way that

$$v_\varepsilon(x) > C_2 \geqslant u_\varepsilon(x), \quad |x| = \delta_0, \quad 0 < \varepsilon \leqslant \varepsilon_0; \tag{4.16}$$

we may assume that δ_0 is so small that $\sqrt{C_1}\delta_0$ is smaller than the first zero of J_0, where C_1 is as in part (1) of the present proof, and that there is a constant C_3 such that $|\alpha_\varepsilon| \leqslant C_3$, for $0 < \varepsilon \leqslant \varepsilon_1$. Notice that v_ε is smooth in Ω (up to the boundary) and satisfies $-\Delta v_\varepsilon = \lambda_1(\varepsilon) v_\varepsilon$.

We are going to show that the positive part of $u_\varepsilon - v_\varepsilon$ has L_2-norm zero over $B_{\delta_0} \backslash K_\varepsilon$, from which the desired result follows. We let $w_\varepsilon = u_\varepsilon - v_\varepsilon$ and

$$\tilde{w}_\varepsilon^+(x) = \begin{cases} \max\{u_\varepsilon - v_\varepsilon, 0\}, & x \in B_{\delta_0} \backslash K_\varepsilon, \\ 0, & x \in \Omega \backslash B_{\delta_0}, \end{cases} \tag{4.17}$$

clearly, $\tilde{w}_\varepsilon^+ \in \mathscr{H}^1_{\mathrm{DN}}(\Omega_\varepsilon)$. We first wish to perform an integration by parts in the expression $\langle -\Delta w_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle$. By the definition of H_ε given in (1.8) we have

$$\langle -\Delta u_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle = \langle H_\varepsilon u_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle = \langle \nabla u_\varepsilon\,,\ \nabla \tilde{w}_\varepsilon^+\rangle. \tag{4.18}$$

Next, writing $W_{n,\varepsilon} = B_{\delta_0} \backslash (\varepsilon \overline{G}_n)$,

$$\langle -\Delta v_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle = \int_{\Omega_\varepsilon} (-\Delta v_\varepsilon) \tilde{w}_\varepsilon^+ \,\mathrm{d}x = \lim_{n \to \infty} \int_{W_{n,\varepsilon}} (-\Delta v_\varepsilon) \tilde{w}_\varepsilon^+ \,\mathrm{d}x, \tag{4.19}$$

by dominated convergence. Applying Green's formula on $W_{n,\varepsilon}$ (and noting that $\tilde{w}_\varepsilon^+$ vanishes identically in a neighborhood of ∂B_{δ_0}), we find

$$\int_{W_{n,\varepsilon}} (-\Delta v_\varepsilon)\tilde{w}_\varepsilon^+ \,\mathrm{d}x = \langle \nabla v_\varepsilon\,,\ \nabla \tilde{w}_\varepsilon^+\rangle_{L_2(W_{n,\varepsilon})} - \int_{\varepsilon\partial G_n} (\partial_\nu v_\varepsilon)\tilde{w}_\varepsilon^+ \,\mathrm{d}\omega, \tag{4.20}$$

where the normal ν at $x \in \partial G_n$ is the interior normal of G_n.

Now v_ε being a radial, decreasing function in B_{δ_0}, Assumption III implies that $\partial_\nu v_\varepsilon$ is non-negative. Taking the limit $n \to \infty$ in (4.20), we obtain

$$\langle -\Delta v_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle \leqslant \langle \nabla v_\varepsilon\,,\ \nabla \tilde{w}_\varepsilon^+\rangle. \tag{4.21}$$

Combining (4.21) with (4.18) we see that

$$\langle -\Delta w_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle \geqslant \langle \nabla w_\varepsilon\,,\ \nabla \tilde{w}_\varepsilon^+\rangle = \|\nabla \tilde{w}_\varepsilon^+\|^2 \geqslant c_1 \delta_0^{-2} \|\tilde{w}_\varepsilon^+\|^2, \tag{4.22}$$

by part (1) of the present proof. We now conclude that

$$C_1 \|\tilde{w}_\varepsilon^+\|^2 \geqslant \lambda(\varepsilon)\|\tilde{w}_\varepsilon^+\|^2 = \langle \lambda_1(\varepsilon) w_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle = \langle -\Delta w_\varepsilon\,,\ \tilde{w}_\varepsilon^+\rangle \geqslant c_1\delta_0^{-2}\|\tilde{w}_\varepsilon^+\|^2. \tag{4.23}$$

But $c_1\delta_0^{-2} > C_1$ and hence $\tilde{w}_\varepsilon^+ = 0$, for $0 < \varepsilon \leqslant \varepsilon_0$, as desired. $\square$

Proposition 4.1 and Lemma 4.3 yield the following corollary.

Corollary 4.4. *Suppose Ω and K_ε satisfy Assumptions* I–III *with $K_\varepsilon = \varepsilon K_1$, and let $\lambda_1(\varepsilon)$ be as above. Then there exist $\varepsilon_0 > 0$ and $c \geqslant 0$ such that*

$$|\lambda_1(\varepsilon) - \Lambda_1| \leqslant c\varepsilon^d, \quad 0 < \varepsilon < \varepsilon_0. \tag{4.24}$$

5. Appendix: Asymptotics of Bessel functions

In this appendix, we will study linear combinations of Bessel functions [22,35] that satisfy a Neumann boundary condition at $r = \varepsilon$, for $\varepsilon > 0$. We begin with simple upper and lower bounds for J_n, J_n', Y_n, and Y_n' at small arguments where the dependence of the constants on $n \in \mathbb{N}$ is made explicit.

Lemma 5.1. *For $n \in \mathbb{N}$, let J_n, Y_n denote the Bessel functions of the first and second kind (the functions Y_n are also called Weber or Neumann functions). Then there exist positive constants r_0 and $c_1, \ldots, c_4$ that are independent of $n \in \mathbb{N}$ such that*

$$\frac{1}{2n!}(r/2)^n \leqslant |J_n(r)| \leqslant \frac{2}{n!}(r/2)^n, \quad 0 < r \leqslant 1, \tag{5.1}$$

$$\frac{1}{8(n-1)!}(r/2)^{n-1} \leqslant |J_n'(r)| \leqslant \frac{2}{(n-1)!}(r/2)^{n-1}, \quad 0 < r \leqslant 1, \tag{5.2}$$

$$c_1(n-1)!(r/2)^{-n} \leqslant |Y_n(r)| \leqslant c_2(n-1)!(r/2)^{-n}, \quad 0 < r \leqslant r_0, \tag{5.3}$$

$$c_3 n!(r/2)^{-n-1} \leqslant |Y_n'(r)| \leqslant c_4 n!(r/2)^{-n-1}, \quad 0 < r \leqslant r_0. \tag{5.4}$$

Proof. In the following proof, we always assume $r \in (0, 1]$, if not otherwise specified. From the power series for J_n we immediately obtain

$$\left| J_n(r) - \frac{1}{n!}(r/2)^n \right| \leqslant \frac{1}{(n+1)!}(r/2)^{n+2},$$

and (5.1) follows. The functional equation $J_n' = \frac{n}{r} J_n - J_{n+1}$ and (5.1) yield

$$|J_n'(r)| \leqslant \frac{1}{(n-1)!}(r/2)^{n-1} + \frac{2}{(n+1)!}(r/2)^{n+1} \leqslant \frac{3}{2(n-1)!}(r/2)^{n-1}$$

and

$$\begin{aligned} |J_n'(r)| \geqslant |\frac{n}{r} J_n(r)| - |J_{n+1}(r)| &\geqslant \frac{1}{4(n-1)!}(r/2)^{n-1} - \frac{2}{(n+1)!}(r/2)^{n+1} \\ &\geqslant \left(\frac{1}{4(n-1)!} - \frac{1}{8(n-1)!} \right) (r/2)^{n-1}, \end{aligned}$$

and (5.2) follows.

For the functions Y_n we use the standard power series expansion [31, p. 310] of the form $Y_n(r) = E_1^{(n)}(r) - E_2^{(n)}(r) - E_3^{(n)}(r)$, with

$$\begin{aligned} E_1^{(n)}(r) &= \frac{2}{\pi} J_n(r) \left(\log \frac{r}{2} + \gamma \right), \\ E_2^{(n)}(r) &= \frac{1}{\pi}(r/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(r/2)^{2k} \left(\sum_{j=1}^{k} \frac{1}{j} + \sum_{j=1}^{n+k} \frac{1}{j} \right), \\ E_3^{(n)}(r) &= \frac{1}{\pi}(r/2)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(r/2)^{2k}, \end{aligned}$$

here γ denotes Euler's constant. By (5.1), there exists a constant $C_3 \geqslant 0$ such that $|E_1^{(n)}(r)| \leqslant C_3$, for all $n \in \mathbb{N}$. Next, it is easy to see that there is a constant C_4 such that $|E_2^{(n)}(r)| \leqslant C_4$, for all $n \in \mathbb{N}$. From the polynomial expression $E_3^{(n)}(r)$ we keep the term $\frac{1}{\pi}(n-1)!(r/2)^{-n}$ while we use

$$0 \leqslant \sum_{k=1}^{n-1} \frac{(n-k-1)!}{k!}(r/2)^{2k} \leqslant \mathrm{e}(n-2)!(r/2)^2, \quad n \geqslant 2,$$

to estimate the remaining terms in $E_3^{(n)}$. It follows that

$$\left| Y_n(r) + \frac{1}{\pi}(n-1)!(r/2)^{-n} \right| \leqslant C_5(n-2)!(r/2)^{2-n} + C_6, \quad n \geqslant 2, \tag{5.5}$$

and we obtain (5.3). Using the functional equation for Y_n', we get from (5.5)

$$
\begin{aligned}
|Y_n'(r)| &\geqslant \left|\frac{n}{r}Y_n(r) - Y_{n+1}(r)\right| \\
&\geqslant 2\frac{n!}{\pi}(r/2)^{-n-1} - \frac{n!}{\pi}(r/2)^{-n-1} \\
&\quad - C_7\left(\frac{n!}{n-1} + (n-1)!\right)(r/2)^{-n+1} - C_8\left(1+\frac{n}{r}\right).
\end{aligned}
$$

It follows that there exist positive constants r_0 and C_9 such that

$$|Y_n'(r)| \geqslant C_9 n!(r/2)^{-n-1}, \quad 0 < r \leqslant r_0, \quad n \in \mathbb{N}.$$

The upper estimate for $Y_n'(r)$ is obtained in a similar fashion and (5.4) follows. □

Remark. This lemma generalizes easily to the higher dimensional Bessel functions as discussed, e.g., in [22]. A starting point for the estimates is provided by [22, p. 122, Lemma 3].

We now apply the above estimates to linear combinations of Bessel functions satisfying a Neumann boundary condition at $r = \varepsilon$.

Lemma 5.2. *Let $\delta \in (0, \min\{1, r_0\}]$ be fixed, where r_0 is as in Lemma 5.1, and let $0 < \varepsilon \leqslant \delta$. For $n \in \mathbb{N}$, let J_n, Y_n be as above and define*

$$f_{n;\varepsilon} = f_{n;\varepsilon}(r) = J_n(r) + \beta_n Y_n(r), \quad n \in \mathbb{N},$$

with $\beta_n = \beta_n(\varepsilon) \in \mathbb{R}$ chosen in such a way that $f_{n;\varepsilon}'(\varepsilon) = 0$.

Then there exist constants $C_0 > 0$ and $\varepsilon_0 = \varepsilon_0(\delta) > 0$, independent of n, such that

$$|f_{n;\varepsilon}(\varepsilon)| \leqslant C_0 \frac{\varepsilon^n}{\delta^n} |f_{n;\varepsilon}(\delta)|, \quad n \in \mathbb{N}, \quad 0 < \varepsilon < \varepsilon_0.$$

In particular, $f_{n;\varepsilon}(\delta) \neq 0$, for all $n \in \mathbb{N}$.

Proof. From the condition $J_n'(\varepsilon) + \beta_n Y_n'(\varepsilon) = 0$ we find

$$\beta_n = -J_n'(\varepsilon)/Y_n'(\varepsilon)$$

(where we also note that, for all $n \in \mathbb{N}$, Y_n' does not vanish in $(0, r_0)$ by Lemma 5.1). As $f_{n;\varepsilon}$ has only isolated zeros, we may assume without loss of generality that $f_{n;\varepsilon}(\delta) \neq 0$ and compute

$$
\begin{aligned}
\frac{f_{n;\varepsilon}(\varepsilon)}{f_{n;\varepsilon}(\delta)} &= \frac{J_n(\varepsilon)Y_n'(\varepsilon) - J_n'(\varepsilon)Y_n(\varepsilon)}{J_n(\delta)Y_n'(\varepsilon) - J_n'(\varepsilon)Y_n(\delta)}, \\
&= \frac{2/\pi\varepsilon}{J_n(\delta)Y_n'(\varepsilon) - J_n'(\varepsilon)Y_n(\delta)},
\end{aligned}
\tag{5.6}
$$

by the formula for the Wronskian of J_n and Y_n [35, p. 76]; note that the Wronskian is independent of n. In the denominator of (5.6) we apply the estimates provided in Lemma 5.1 to obtain

$$|J_n(\delta)Y_n'(\varepsilon)| \geqslant c_1 \varepsilon^{-n-1}\delta^n, \quad |J_n'(\varepsilon)Y_n(\delta)| \leqslant c_2 \varepsilon^{n-1}, \tag{5.7}$$

with constants c_1, c_2 independent of n. Plugging (5.7) into (5.6) and choosing $\varepsilon_0 > 0$ such that $c_2\varepsilon_0^2\delta^{-1} \leqslant c_1/2$ we arrive at the desired result. □

The proof of Lemma 5.2 easily extends to cover a slightly more general result where we consider $J_n(\mu r)$ and $Y_n(\mu r)$ with a parameter μ from a compact subset of $(0, \infty)$. In the following lemma, we let $\varepsilon_1 > 0$ be such that $\lambda_1(\varepsilon) \in M := [\Lambda_1/2, 2\Lambda_1]$, for $0 < \varepsilon \leqslant \varepsilon_1$. As $\lambda_1(\varepsilon) \to \Lambda_1$ by Theorem 2.4 such an $\varepsilon_1 > 0$ exists. Furthermore, it follows from (5.3) that there exists $\delta_1 > 0$ such that $Y_n'(\mu r) \neq 0$ for all $n \in \mathbb{N}$, $\mu \in M$ and $0 < r \leqslant \delta_1$.

Lemma 5.3. *Let* ε_1, δ_1 *as above, and let* J_n, Y_n *as in Lemma* 5.1. *Let* $0 < \delta < \min\{1, \delta_1\}$ *be fixed and let*

$$f_{n;\varepsilon,\mu} = f_{n;\varepsilon,\mu}(r) = J_n(\mu r) + \beta_n Y_n(\mu r), \quad n \in \mathbb{N}_0, \quad 0 < \varepsilon \leqslant \varepsilon_1,$$

with $\beta_n = \beta_n(\varepsilon, \mu) \in \mathbb{R}$ *chosen in such a way that* $f_{n;\varepsilon,\mu}'(\varepsilon) = 0$.

Then there exist constants $C_0 \geqslant 0$ *and* $\varepsilon_0 > 0$ *such that*

$$|f_{n;\varepsilon,\mu}(\varepsilon)| \leqslant C_0 \frac{\varepsilon^n}{\delta^n} |f_{n;\varepsilon,\mu}(\delta)| \quad n \in \mathbb{N}, \quad 0 < \varepsilon < \varepsilon_0, \quad \mu \in M.$$

Furthermore, $f_{n;\varepsilon,\mu}(\delta) \neq 0$, *for all* $n \in \mathbb{N}$, $0 < \varepsilon < \varepsilon_0$, *and* $\mu \in M$.

The proof is analogous to that of Lemma 5.2 and is omitted.

We finally produce a simple lower bound for the lowest eigenvalue $\lambda_1(B_r, \overline{B}_\varrho)$ of the mixed boundary value problem on an annulus or a spherical shell with radii $0 < \varrho < r$ with Dirichlet boundary condition at $|x| = r$ and Neumann boundary condition at $|x| = \varrho$. The corresponding Laplacian is denoted as $H(B_r, \overline{B}_\varrho)$, cf. Section 1.

Lemma 5.4. *There exists a positive constant*, c, *such that*

$$\lambda_1(B_r, \overline{B}_\varrho) \geqslant c/r^2, \quad 0 < \varrho < r.$$

Proof. (Only given for $d = 2$). By scaling, it is enough to obtain a constant $c_1 > 0$ such that $\lambda_1(B_1, \overline{B}_\varrho) \geqslant c_1$ for $0 < \varrho \leqslant 1$. $\lambda_1(B_1, \overline{B}_\varrho)$ being a continuous and positive function of $\varrho \in (0, 1]$, we only have to exclude that $\lambda_1(B_1, \overline{B}_\varrho) \to 0$ as $\varrho \to 0$. Let $\mu = \mu(\varrho)$ denote the (positive) square root of $\lambda_1(B_1, \overline{B}_\varrho)$ and assume for a contradiction that $\mu(\varrho) \to 0$ as $\varrho \to 0$. Also let f_ϱ denote an eigenfunction of $H(B_1, \overline{B}_\varrho)$ for the eigenvalue $\mu(\varrho)^2$. With J_0, Y_0 denoting the Bessel functions of the first and second kind, respectively, f_ϱ can be written as

$$f_\varrho(r) = \alpha(J_0(\mu r) + \beta Y_0(\mu r)), \quad 0 < r \leqslant 1,$$

with boundary conditions $f_\varrho(1) = 0 = f_\varrho'(\varrho)$; for simplicity, we write $\mu = \mu(\varrho)$ and $\beta = \beta(\varrho)$. The boundary conditions translate into $J_0(\mu) + \beta Y_0(\mu) = 0$ and $J_0'(\mu\varrho) + \beta Y_0'(\mu\varrho) = 0$. There is $r_0 > 0$ such that $Y_0'(r) \neq 0$, for $0 < r \leqslant r_0$, and we may compute $\beta = -J_0'(\mu\varrho)/Y_0'(\mu\varrho)$, for $0 < \varrho \leqslant \varrho_0$, for some $\varrho_0 > 0$. It follows that $J_0(\mu)Y_0'(\mu\varrho) - J_0'(\mu\varrho)Y_0(\mu) = 0$. Here it is easy to see that, as $\varrho \to 0$, $J_0(\mu)Y_0'(\mu\varrho) \to -\infty$ while $J_0'(\mu\varrho)Y_0(\mu) \to 0$, a contradiction. □

Remark. It is easy to see that, in fact, $\beta(\varrho) \to 0$, as $\varrho \to 0$ and hence $\lambda_1(B_1, \overline{B}_\varrho) \to \Lambda_1^{B_1}$, as $\varrho \to 0$.

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